QUANTUM LIMITS OF MEASUREMENT AND COMPUTING INDUCED BY CONSERVATION LAWS AND UNCERTAINTY RELATIONS

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A quantitative extension of the Wigner-Araki-Yanase theorem is obtained on the limitation on precise, non-disturbing measurements of observables which do not commute with additive conserved quantities, and applied to obtaining a limitation on the accuracy of quantum computing with computational bases which do not commute with angular momentum.

1 Introduction

Errors in quantum computers can be classified into two classes: the static errors—decoherence in qubits arising from the interaction between computational qubits and the environment—and the dynamical errors—imperfection of logic operations arising from the interaction between computational qubits and controllers of quantum gates. The current theory of fault-tolerant quantum computing concludes that if the imperfection is below a certain threshold, the decoherence can be corrected in an arbitrarily large quantum computing.[1] Thus, the fundamental question as to whether quantum computers are physically realizable or not can be reduced to such questions as whether fundamental physical laws lead to an unavoidable imperfection of quantum logic operations or not. On the other hand, it has been known in measurement theory that conservation laws limit the accuracy of measurements, as stated by the Wigner-Araki-Yanase (WAY) theorem:[2,3,4,5] Observables which do not commute with bounded additive conserved quantities have no precise, non-disturbing measurements. During computation, a quantum computer usually needs to "measure" internal qubits to perform the branching program. Such "measurements" are carried out by the so-called controlled-not (CNOT) gates. Thus, it is natural to ask whether the WAY theorem leads to a conflict between the accuracy of quantum logic operations and conservation laws.

In this paper, we give a quantitative expression of the WAY theorem to obtain a lower bound on the error and the disturbance. Then, we apply this bound to the implementations of CNOT gates and obtain the following results: If the computational basis is represented by a component of spin and physical implementations obey the angular momentum conservation law, any physically realizable unitary operations on the two computational qubits plus the $S-2$ ancilla qubits cannot implement the CNOT gate within the error probability $1/(4S^2)$. An analogous relation for the bosonic control field is also obtained to show that the error probability has the lower bound $1/(16\bar{n})$, where $\bar{n}$ is the average photon number of the control field. From the above result, we can conclude that in order to avoid the entanglement between computational qubits and the controlling system, we need to use the computational basis which commute with all conserved quantities, or to use a very large ancilla or a very strong control field to the extent described by the above limitation. The former conclusion is closely related to the proposal of "encoded universality" [6,7] and the latter is closely related to the recent research on imperfect operations caused by the entanglement between qubits and the controlling laser field.[8,9] The present result will give a unified basis for those researches on quantum computing, providing with rigorous and general, quantitative methods.

2 Error and Disturbance in Quantum Measurement

Let $\mathbf{M}(\mathbf{x})$ be a measuring apparatus with macroscopic output variable $\mathbf{x}$ to measure, possibly with some error, an observable A of the *object* $\mathbf{S}$, a quantum system represented by a Hilbert space $\mathcal{H}_{\mathbf{S}}$. The measuring interaction turns on at time 0 and turns off at time Δt between the object $\mathbf{S}$ and the measuring apparatus $\mathbf{M}(\mathbf{x})$. We assume that the apparatus has two parts, the *probe* $\mathbf{P}$ represented by a Hilbert space $\mathcal{H}_{\mathbf{P}}$ and the *ancilla* $\mathbf{A}$ represented by a Hilbert space $\mathcal{H}_{\mathbf{A}}$, and that the composite system $\mathbf{S}+\mathbf{P}+\mathbf{A}$ evolves unitarily from time 0 to time Δt. Denote by U the unitary operator on $\mathcal{H}_{\mathbf{S}} \otimes \mathcal{H}_{\mathbf{P}} \otimes \mathcal{H}_{\mathbf{A}}$ representing the time evolution of $\mathbf{S}+\mathbf{P}+\mathbf{A}$ in the time interval $(0, \Delta t)$.

At time 0, the object, the probe, and the ancilla are supposed to be in states ψ, ϕ, and ξ, respectively; all state vectors are assumed to be normalized unless stated otherwise. Thus, the composite system $\mathbf{S}+\mathbf{P}+\mathbf{A}$ is in the state $\psi\otimes\phi\otimes\xi$ at time 0. Just after the measuring interaction turns off, the probe is subjected to a local interaction with the subsequent stages of the apparatus. The last process is assumed to precisely measure an observable M in the probe $\mathbf{P}$ and to output the result of the measurement as the value of the macroscopic outcome variable $\mathbf{x}$. The statistical properties of the apparatus $\mathbf{M}(\mathbf{x})$ is then determined by the given quadruple $(\mathcal{H}_{\mathbf{P}} \otimes \mathcal{H}_{\mathbf{A}}, \phi \otimes \xi, U, M)$, which is called the indirect measurement model of $\mathbf{M}(\mathbf{x})$.[10,11] In the Heisenberg picture with the original state $\psi\otimes\phi\otimes\xi$, we shall write $A(0) = A\otimes I\otimes I$, $M(0) = I\otimes M\otimes I$, $A(\Delta t) = U^{\dagger}(A \otimes I \otimes I)U$, and $M(\Delta t) = U^{\dagger}(I \otimes M \otimes I)U$.

We say that measuring apparatus $\mathbf{M}(\mathbf{x})$ *measures* observable A *precisely*, if $A(0)$ and $M(\Delta t)$ have the same probability distribution on any input state ψ. The *error operator* $E(A)$ of apparatus $\mathbf{M}(\mathbf{x})$ for measuring A is defined by $E(A) = M(\Delta t) - A(0)$. The *(root-mean-square) error* $\epsilon(A)$ of apparatus $\mathbf{M}(\mathbf{x})$ for measuring A on input state ψ is, then, defined by $\epsilon(A) = \langle E(A)^2\rangle^{1/2}$, where $\langle \cdots \rangle$ stands for $\langle \psi \otimes \phi \otimes \xi | \cdots | \psi \otimes \phi \otimes \xi \rangle$. Then, we have

$$\epsilon(A)^2 = \sigma[E(A)]^2 + \langle E(A) \rangle^2 \geq \sigma[E(A)]^2, \tag{1}$$

where $\sigma(X)$ stands for the standard deviation of an observable X in $\psi \otimes \phi \otimes \xi$, i.e., $\sigma(X)^2 = \langle X^2 \rangle - \langle X \rangle^2$. It can be shown that apparatus $\mathbf{M}(\mathbf{x})$ measures observable A precisely if and only if $\epsilon(A) = 0$ for any input state ψ.

The apparatus $\mathbf{M}(\mathbf{x})$ is called *non-disturbing*, if $A(0)$ and $A(\Delta t)$ have the same probability distribution on any input state ψ. The *disturbance operator* $D(A)$ of apparatus $\mathbf{M}(\mathbf{x})$ for observable A is defined by $D(A) = A(\Delta t) - A(0)$. The *(root-mean-square) disturbance* $\eta(A)$ of apparatus $\mathbf{M}(\mathbf{x})$ for observable A on input state ψ is, then, defined by $\eta(A) = \langle D(A)^2 \rangle^{1/2}$. Then, we have

$$\eta(A)^2 = \sigma[D(A)]^2 + \langle D(A) \rangle^2 \geq \sigma[D(A)]^2. \tag{2}$$

It can be shown that apparatus $\mathbf{M}(\mathbf{x})$ is non-disturbing if and only if $\eta(A) = 0$ for any input state ψ.

3 Quantitative Generalizations of Wigner-Araki-Yanase Theorem

Assume the additive conservation law (ACL) on quantities L_1 of the object $\mathbf{S}$, L_2 of the probe $\mathbf{P}$, and L_3 of the ancilla $\mathbf{A}$. In the Heisenberg picture, we shall write $L_1(0) = L_1 \otimes I \otimes I$, $L_1(\Delta t) = U^\dagger (L_1 \otimes I \otimes I) U$, $L_2(0) = I \otimes L_2 \otimes I$, $L_2(\Delta t) = U^\dagger (I \otimes L_2 \otimes I) U$, and so on. Then, the ACL is formulated as

$$L_1(0) + L_2(0) + L_3(0) = L_1(\Delta t) + L_2(\Delta t) + L_2(\Delta t). \tag{3}$$

From Eq. (3), we have the following commutation relations

$$[A(0), L_1(0)] = [L_1(\Delta t), E(A)] + [L_2(\Delta t), D(A)] + [L_3(\Delta t), D(A)], \tag{4}$$

$$[A(0), L_1(0)] = [L_1(\Delta t), E(A)] + [L_2(\Delta t), D(A)] + [L_3(\Delta t), E(A)]. \tag{5}$$

Taking the modulus of the expectations of the both sides of Eq. (4) and applying the triangular inequality, we have

$$|\langle [A(0), L_1(0)] \rangle| \leq |\langle [L_1(\Delta t), E(A)] \rangle| + |\langle [L_2(\Delta t), D(A)] \rangle| + |\langle [L_3(\Delta t), E(A)] \rangle|. \tag{6}$$

By the Robertson uncertainty relation (i.e., $\sigma(X)\sigma(Y) \geq |\langle [X, Y] \rangle|/2$ for any observables X and Y), we have

$$\frac{1}{2} |\langle \psi | [A, L_1] | \psi \rangle| \leq \epsilon(A)\sigma[L_1(\Delta t)] + \eta(A)\sigma[L_2(\Delta t)] + \eta(A)\sigma[L_3(\Delta t)]. \tag{7}$$

Similarly, from Eq. (5), we obtain

$$\frac{1}{2} |\langle \psi | [A, L_1] | \psi \rangle| \leq \epsilon(A)\sigma[L_1(\Delta t)] + \eta(A)\sigma[L_2(\Delta t)] + \epsilon(A)\sigma[L_3(\Delta t)]. \tag{8}$$

Each of the above inequalities, (7) and (8), implies the WAY theorem as follows. If the conserved quantities are bounded, we have

$\sigma[L_1(\Delta t)], \sigma[L_2(\Delta t)], \sigma[L_3(\Delta t)] < \infty$. Thus, if the measurement is precise and no-disturbing, i.e., $\epsilon(A) = \eta(A) = 0$ for any state ψ, then we have $[A, L_1] = 0$. Thus, we conclude that observables which do not commute with bounded additive conserved quantities allow no precise, non-disturbing measurement.

Summing up both of the above inequality and using the inequality $ax + by \le \max\{a, b\}(x + y)$ for $a, b, x, y \ge 0$, we have

$$|\langle\psi|[A, L_1]|\psi\rangle| \le [\epsilon(A) + \eta(A)](2 \max\{\sigma[L_1(\Delta t)], \sigma[L_2(\Delta t)]\} + \sigma[L_3(\Delta t)]). \tag{9}$$

Since $\sigma(X) \le \|X\| = \|U^\dagger X U\|$ for any observable X and any unitary operator U, by the inequality $(x + y)^2 \le 2(x^2 + y^2)$ we have

$$\frac{|\langle\psi|[A, L_1]|\psi\rangle|^2}{2(2 \max\{\|L_1\|, \|L_2\|\} + \sigma[L_3(\Delta t)])^2} \le \epsilon(A)^2 + \eta(A)^2. \tag{10}$$

4 Physical Implementations of CNOT Gates

Let U_{CN} be a CNOT gate on a 2-qubit system $\mathbf{C} + \mathbf{T}$. Let X_i, Y_i, and Z_i be the Pauli operators of qubit $\mathbf{C}$ for $i = 1$ or qubit $\mathbf{T}$ for $i = 2$ defined by $X_i = |1\rangle\langle 0| + |0\rangle\langle 1|$, $Y_i = i|1\rangle\langle 0| - i|0\rangle\langle 1|$, and $Z_i = |0\rangle\langle 0| - |1\rangle\langle 1|$ with the computational basis $\{|0\rangle, |1\rangle\}$. On the computational basis, U_{CN} acts as $U_{\mathrm{CN}}|a, b\rangle = |a, b \oplus a\rangle$ for $a, b = 0, 1$, where $\oplus$ denotes the addition modulo 2. Thus, in particular, we have $U_{\mathrm{CN}}|a, 0\rangle = |a, a\rangle$ for $a = 0, 1$. The above relation shows that the unitary operator U_{CN} serves as an interaction between the "object" $\mathbf{C}$ and the "probe" $\mathbf{T}$ for a precise, non-disturbing measurement of Z_1 with probe observable Z_2 without ancilla system. Thus, by the WAY theorem, if there are additive conserved quantities not commuting with Z_1, the unitary operator U_{CN} cannot be implemented correctly.

Let $\alpha = (U, |\xi\rangle)$ be a physical implementation of U_{CN} defined by a unitary operator U on the system $\mathbf{C}+\mathbf{T}+\mathbf{A}$, where $\mathbf{A}$ is the *ancilla*, and a state vector $|\xi\rangle$ of the ancilla. The implementation $\alpha = (U, |\xi\rangle)$ defines a trace-preserving quantum operation $\mathcal{E}_\alpha$ by

$$\mathcal{E}_\alpha(\rho) = \mathrm{T}r_{\mathbf{A}}[U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger] \tag{11}$$

for any density operator ρ of the system $\mathbf{C} + \mathbf{T}$, where $\mathrm{T}r_{\mathbf{A}}$ stands for the partial trace over the system $\mathbf{A}$. The *gate fidelity* [1] of α is defined by

$$F(\mathcal{E}_\alpha, U_{\mathrm{CN}}) = \min_{|\psi\rangle} F(\psi) \tag{12}$$

where $|\psi\rangle$ varies over all state vectors of $\mathbf{C} + \mathbf{T}$, and $F(\psi)$ is the fidelity of two states $U_{CN}|\psi\rangle$ and $\mathcal{E}_\alpha(|\psi\rangle\langle\psi|)$. Then, $1 - F(\mathcal{E}_\alpha, U_{CN})^2$ is a good measure for the worst error probability of the implementation α over all possible input states.[12]

5 Imperfection from Angular Momentum Conservation Law

Let us consider the computational basis defined by the spin component of the z direction and the angular momentum conservation law for the x direction. Thus, we assume $L_i = X_i$ for $i = 1, 2$, so that $\|L_1\| = \|L_2\| = 1$, and that L_3 is considered as the x-component of the total angular momentum divided by $\hbar/2$ of the ancilla system $\mathbf{A}$. Then, the unitary operator U should satisfy the conservation law $[U, L_1 + L_2 + L_3] = 0$. Letting $A = Z_1$ and $M = Z_2$ and applying Eq. (10), we have

$$\frac{|\langle\psi|[Z_1, X_1]|\psi\rangle|^2}{2[2+\sigma(L_3')]^2} \le \epsilon(Z_1)^2 + \eta(Z_1)^2, \tag{13}$$

where $L_3' = U^\dagger(I \otimes I \otimes L_3)U$.

It can be shown that $U = U_{\mathrm{CN}}$ on $\mathcal{H}_{\mathbf{T}} \otimes \mathcal{H}_{\mathbf{C}}$ if and only if $\epsilon(Z_1)^2 + \eta(Z_1)^2 = 0$ holds for $\phi = \langle 0\rangle$ and any ψ, ξ. Thus, $\epsilon(Z_1)^2 + \eta(Z_1)^2$ measures the imperfection of U in implementing U_{CN}. Relation (13) suggests that in order to make U more accurately implement U_{CN} we need to make $[Z_1, X_1]$ smaller or make $\sigma(L_3')$ larger.

In order to relate the measure $\epsilon(Z_1)^2 + \eta(Z_1)^2$ to the gate fidelity and to obtain an upper bound for the gate fidelity, assume $\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\phi = |0\rangle$. Then, we have

$$\epsilon(Z_1)^2 + \eta(Z_1)^2 \le 8[1 - F(\mathcal{E}_\alpha, U_{\mathrm{CN}})^2] \tag{14}$$

for any ξ.[12] Since $[Z_1, L_1] = [Z_1, X_1] = 2iY_1$, we have

$$|\langle [Z_1, X_1]\rangle| = 2. \tag{15}$$

Thus, from Eqs. (13)–(15), we have the following fundamental upper bound of the gate fidelity

$$F(\mathcal{E}_\alpha, U_{\mathrm{CN}})^2 \le 1 - \frac{1}{4[2+\sigma(L_3')]^2}. \tag{16}$$

In the following, we shall interpret the above relation in terms of the notion of the size of implementations for fermionic and bosonic ancillae separately.

We now assume that the ancilla $\mathbf{A}$ comprises qubits. Then, the size $s(\alpha)$ of the implementation α is defined to be the total number n of the qubits included in $\mathbf{C} + \mathbf{T} + \mathbf{A}$. Then, we have $\Delta L_3' \le \|L_3\| = n - 2$. Thus, we have the following upper bound of the gate fidelity

$$F(\mathcal{E}_\alpha, U_{\mathrm{CN}})^2 \le 1 - \frac{1}{4s(\alpha)^2}, \tag{17}$$

with $s(\alpha) = n$. Therefore, it has been proven that if the computational basis is represented by the z-component of spin, any implementation with size n which preserves the x-component of angular momentum cannot implement the CNOT gate within the error probability $1/(4n^2)$. In particular, any implementation on $\mathbf{C}+\mathbf{T}$ cannot simulate U_{CN} within the error probability 1/16.

In current proposals,[1] the external electromagnetic field prepared by the laser beam is considered to be a feasible candidate for the ancilla $\mathbf{A}$ to be coupled with the computational qubits $\mathbf{C}+\mathbf{T}$ via the dipole interaction. Then, the ancilla state ξ is considered to be a coherent state, for which we have $(\Delta N)^2 = \langle\xi|N|\xi\rangle = \langle N\rangle$, where N is the number operator. We assume that the beam propagates to the x-direction with right-hand circular polarization. Then, we have $L_3 = 2N$, and hence $\Delta L_3' = 2\Delta N' = 2\langle N'\rangle^{1/2} \le 2(\langle N\rangle + 2)^{1/2}$. Thus, Eq. (17) holds with defining the size of implementation α by $s(\alpha) = 2\langle N\rangle^{1/2}$ appropriately for the strong field, and hence Eq. (17) turns to be the relation

$$F(\mathcal{E}_\alpha, U_{\mathrm{CN}})^2 \le 1 - \frac{1}{16\langle N\rangle}. \tag{18}$$

Therefore, the lower bound of the error probability is inversely proportional to the average number of photons. This bound is consistent with the recent calculation on error probability based on a "one-photon type" transition Hamiltonian.[9]

In this paper, we have concentrated on CNOT gates, for discussions on other logic operations we refer to the recent publication.[12]

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